3rd Annual Lexington Mathematical Tournament
Individual Round

Solutions

1. Answer: $\boxed{153}$

   Solution: $1! + 2! + 3! + 4! + 5! = 1 + 2 + 6 + 24 + 120 = 153$.

2. Answer: $\boxed{\frac{22}{25}}$

   Solution: There are a total of 25 jelly beans, and 22 of them are not spinach. Therefore, the answer is $\frac{22}{25}$.

3. Answer: $\boxed{63}$

   Solution: The factors are 1, 2, 4, 8, 16, and 32, which have a total sum of 63.

4. Answer: $\boxed{\frac{25}{8}}$

   Solution: Let Carol's shadow be x feet long. Then, the flag's shadow is $x + 10$ feet long. Since the ratio of height and length of shadow if the same for Carol and for the pole, we have $\frac{21}{10+x} = \frac{5}{x}$. Cross-multiplying and solving gives $x = \frac{25}{8}$.

5. Answer: $\boxed{1}$

   Solution: We must have $\frac{4}{3}\pi \cdot 7^3 = \frac{4}{3}\pi \cdot 1^3 + \frac{4}{3}\pi \cdot 5^3 + \frac{4}{3}\pi \cdot 6^3 + \frac{4}{3}\pi \cdot x^3 \Rightarrow 7^3 = 1^3 + 5^3 + 6^3 + x^3$. Solving, we get $x = 1$.

6. Answer: $\boxed{0}$

   Solution: By the distance formula, $\sqrt{(2-a)^2 + (-2-0)^2} = \sqrt{(-3-a)^2 + (3-0)^2}$ and $\sqrt{(2-0)^2 + (-2-b)^2} = \sqrt{(-3-0)^2 + (3-b)^2}$. Solving, we get $a = -1$ and $b = 1$, so the answer is 0.

7. Answer: $\boxed{\frac{594}{625}}$

   Solution: The probabilities of getting heads on the first, second, and third turn are $\frac{0}{100}$, $\frac{1}{100}$, and $\frac{4}{100}$. Therefore, the probabilities of getting tails on the first three turns are $\frac{100}{100}$, $\frac{99}{100}$, and $\frac{96}{100}$. The probability that tails is flipped each time is the product of the three probabilities, $\frac{594}{625}$.

8. Answer: $\boxed{3}$

   Solution: If we do not reverse the string, it will take at least four turns to get the V to the left. However, if we do reverse the string, V is automatically at the left. Then, it takes two turns to move the W into its place, so the least number of changes possible is 3.

9. Answer: $\boxed{13}$

   Solution: Since the area inside the square and also inside the rectangle is equal to the area inside the rectangle and also inside the square, we can conclude that the two quadrilaterals have the same area. Then, the area of the square is $\sqrt{169} = 13$.

10. Answer: $\boxed{\text{B, C, A}}$

    Solution: Squaring each number, we get $A^2 = 7500$, $B^2 = 7200$, and $C^2 = 7225$. Therefore, the order of the letters from least to greatest is B, C, A.

11. Answer: $\boxed{630}$

Solution: There are 7! ways to reorder seven different letters. However, since there are three different pairs of identical letters, we must divide our count by $2! \times 2! \times 2! = 8$ to get $\frac{7!}{8} = 630$.

12. Answer: $\boxed{\dfrac{\sqrt{3}}{6}}$

Solution: Let the surface area be A, and let s and t be the length of a side of the cube and the tetrahedron, respectively. We have $6s^2 = A \Rightarrow s = \sqrt{\frac{A}{6}}$ and $4 \cdot t^2 \frac{\sqrt{3}}{4} = A \Rightarrow t = \sqrt{\frac{A}{\sqrt{3}}}$. Then, $r^2 = \frac{s^2}{t^2} = \frac{A/6}{A/(\sqrt{3})} = \frac{\sqrt{3}}{6}$.

13. Answer: $\boxed{3 \text{ or } -4}$

Solution: Let $y = x + \frac{1}{x}$. Then, $x^2 + y + \frac{1}{x^2} = 10$ and $y^2 = x^2 + 2 \cdot x \cdot \frac{1}{x} + \frac{1}{x^2} = x^2 + \frac{1}{x^2} + 2$. Substituting $10 - y$ for $x^2 + \frac{1}{x^2}$, we get $y^2 = 12 - y \Rightarrow (y+4)(y-3) = 0 \Rightarrow x + \frac{1}{x} = y = 3$ or -4.

14. Answer: $\boxed{32 + \dfrac{25}{3}\pi}$

Solution: We look for the volume of the locus of points that are within 1 unit away from the spacecraft. For each face of the craft, we have a region of air just outside the craft, with height 1 and base area equal to the area of the face. These regions have a total volume of $2(1 \cdot 2 \cdot 2) + 4(1 \cdot 2 \cdot 3) = 32$. Then, for each edge, we have an additional region shaped like a quarter of a cylinder with radius 1 and length equal to that of the edge. These regions have a total volume of $8(\frac{1}{4} \cdot \pi(1)^2(2)) + 4(\frac{1}{4} \cdot \pi(1)^2(3)) = 7\pi$. Finally, for each vertex of the craft, we have one eighth of a sphere with radius 1. These regions have a total volume of $8(\frac{1}{8} \cdot \pi(1)^3) = \frac{4}{3}\pi$. Summing everything up, we get $32 + 7\pi + \frac{4}{3}\pi = 32 + \frac{25}{3}\pi$.

15. Answer: $\boxed{\dfrac{100}{7}}$

Solution: We have $AB' = 4$, $B'C = 3$, and $m\angle B = m\angle AB'C = m\angle CB'D = 90$. Let $B'D = x$. Then, by the Pythagorean theorem, $CD = \sqrt{x^2+9}$. By the Pythagorean theorem again, $AB^2 + BD^2 = AD^2 \Rightarrow 4^2 + (3+\sqrt{x^2+9})^2 = (4+x)^2 \Rightarrow 16 + 9 + 6\sqrt{x^2+9} + x^2 + 9 = 16 + 8x + x^2$. Isolating the radical and then squaring gives $36(x^2+9) = 64x^2 - 288x + 324 \Rightarrow 28x^2 = 288x \Rightarrow x = \frac{72}{7}$. Thus, $AD = 4 + \frac{72}{7} = \frac{100}{7}$.

16. Answer: $\boxed{\sqrt{65}}$

Solution: Let the incenter, or center of the incircle, of $\triangle ABD$ be I. Also, let the incenter have radius r. The area of $\triangle ABD$ can be expressed as $\frac{1}{2}(AB)(AD)$ or $\frac{1}{2}(AB)(r) + \frac{1}{2}(BC)(r) + \frac{1}{2}(CA)(r) = \frac{1}{2}Pr$, where P is the perimeter of $\triangle ABD$. (Since the segment connecting the incenter to a tangent point is always perpendicular to the side to which the incircle is tangent, that segment is an altitude to the triangle with vertices at the incenter and at the endpoints of the side.) By the Pythagorean theorem, $BD = 13$, so $P = 5 + 12 + 13 = 30$. Therefore, $\frac{1}{2}(AB)(AD) = \frac{1}{2}(5)(12) = 30 = \frac{1}{2}Pr = 30r$, so $r = 2$. Similarly, the radius of the incircle of $\triangle BCD$ is 2. Thus, the distance between the incircles is $\sqrt{(5-2-2)^2 + (12-2-2)^2} = \sqrt{65}$.

17. Answer: $\boxed{1+\sqrt{2}}$

Solution: We have $ar^3 + ar^2 - ar - a = \frac{1}{2}(ar^4 - a) \Rightarrow a(r^3 + r^2 - r - 1) = \frac{1}{2}a(r^4 - 1) \Rightarrow a(r^2-1)(r+1) = \frac{1}{2}a(r^2-1)(r^2+1) \Rightarrow r + 1 = \frac{1}{2}(r^2+1) \Rightarrow r^2 - 2r - 1 = 0$. By the quadratic formula, the common ratio $r = 1 \pm \sqrt{2} = 1 + \sqrt{2}$ because $r > 0$.
Note: On the day of the tournament, the geometric series was not specified to be increasing. Therefore, 1 was also accepted as a correct answer.

18. Answer: $\boxed{\dfrac{56\sqrt{7}}{11}}$

Solution: Let points E and F be on AB and BC, respectively. Since $\triangle ABC$ and $\triangle EBC$ share the same altitude h_1 from vertex C, $\frac{[EBC]}{[ABC]} = \frac{\frac{1}{2}\cdot EB\cdot h_1}{\frac{1}{2}\cdot AB\cdot h_1} = \frac{EB}{AB} = \frac{4}{9}$ (where $[ABC]$ denotes the area of polygon ABC). Also, $\triangle EBC$ and $\triangle EFC$ share the same altitude h_2 from vertex E, so $\frac{[EFC]}{[EBC]} = \frac{\frac{1}{2}\cdot FC\cdot h_2}{\frac{1}{2}\cdot BC\cdot h_2} = \frac{FC}{BC} = \frac{7}{11}$. Therefore, $\frac{[EBC]}{[ABC]}\cdot\frac{[EFC]}{[EBC]} = \frac{[EBC]}{[ABC]} = \frac{5}{9}\cdot\frac{7}{11} \Rightarrow [EBC] = \frac{28}{99}[ABC]$. By Heron's formula, $[EBC] = \sqrt{s(s-AB)(s-BC)(s=AC)} = \sqrt{18(18-9)(18-16)(18-11)} = 18\sqrt{7}$ (where s is half the perimeter), so $[EBC] = \frac{28}{99}[ABC] = \frac{28}{99}\cdot 18\sqrt{7} = \frac{56\sqrt{7}}{11}$.

19. Answer: $\boxed{288}$

Solution: Let x, y, and z be the rates at which Xavier, Yuna, and Zach run, respectively. Also, let the circumference of the track be d and let time be measured in minutes. Since Zach first passes Xavier in 8 minutes, $8z = 8x + d \Rightarrow z = x + \frac{1}{8}d$. Xavier passes Yuna for the first time in 12 minutes, so $12x = 12y + d \Rightarrow x = y + \frac{1}{12}d$. Substituting x into the first equation, we get $z = y + \frac{1}{12}d + \frac{1}{8}d = y + \frac{5}{24}d \Rightarrow \frac{24}{5}z = \frac{24}{5}y + d$. Therefore, Zach passes Yuna for the first time in $\frac{24}{5}$ minutes, or $\frac{24}{5}\cdot 60 = 288$ seconds.

20. Answer: $\boxed{53}$

Solution: Let N be the six-digit integer that repeats. Then, the unit fraction $\frac{1}{k}$ can be expressed as $\frac{N}{10^6} + \frac{N}{10^{12}} + \frac{N}{10^{18}} + \cdots$. This is a geometric series with common ratio $\frac{1}{10^6}$, which can be expressed as $\frac{N/10^6}{1-1/10^6} = \frac{N}{999999}$. Therefore, $k = \frac{999999}{N}$. For k to be an integer, N must divide (or be a factor of) 999999. Since N is an integer, k must divide 999999. The prime factorization of 999999 is $3^3\cdot 7\cdot 11\cdot 13\cdot 37$, so 999999 has $(3+1)(1+1)(1+1)(1+1)(1+1) = 64$ factors. However, we also know that $\frac{1}{k}$ cannot repeat in 1, 2, or 3 digits, which means that $\frac{1}{k} \neq \frac{M}{9}, \frac{M}{99}, \frac{M}{999}$ for any integer M. It follows that k cannot divide 9, 99, or 999. $999 = 3^4\cdot 37$ has $(3+1)(1+1) = 8$ factors, and $99 = 3^2\cdot 11$ has $(2+1)(1+1) = 6$ factors, but out of these, 1, 3, and 9 are already counted. All factors of 9 are already counted as well. Therefore, k cannot be $8+6-3 = 11$ factors, leaving $64 - 11 = 53$ possible values, so we have 53 possible unit fractions.